GENERAL RELATIVISTIC ROTATIONAL PROPERTIES OF STELLAR MODELS

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Abstract. In this paper we give general relativistic expressions for the angular momentum and rotational kinetic energy of slowly rotating stars. These expressions contain contributions from the presure, gravitational red shift, and Doppler shift, and the motion of inertial frames. These contributions are not negligible, e.g., there are stable neutron star models for which the angular velocity of inertial frames at the center is about 70% the angular velocity of the star. These expressions are useful in the study of pulsars if pulsars are rotating neutron stars.

1. Introduction

In a series of papers, a method for treating slowly rotating fully general relativistic bodies is developed.* In particular, the problem of a slowly rotating fully relativistic spherical shell (originally considered by Thirring (1918) in the weak field limit) was treated. The solution, valid even for strong gravitational fields, exhibits a dragging along of the inertial frames by the shell. In the limit, as the mass (stress-energy) of the shell becomes large compared to the other masses in the universe, the angular velocity of the inertial frames becomes equal to that of the shell (Brill and Cohen, 1966; Cohen, 1965, 1967a, 1968a).

This method for treating rotating bodies in general relativity is also useful in astrophysics. For example, it allows a fully relativistic treatment of rotating stellar models and has been applied to such problems by various authors, e.g., Cohen and Brill (1968), Hartle and Thorne (1968).

It has been suggested that pulsars are rotating neutron stars (Gold, 1968) and that the loss of rotational kinetic energy of the pulsar in the Crab nebulae is the energy source for the Crab (Finzi and Wolf, 1968).

Because typical neutron stars are quite dense (central density $\sim 10^{15}$ gm/cm^3), a full general relativistic treatment is necessary. This is true for the equilibrium structure calculations and the dynamical stability calculations. General relativistic terms also contribute to the angular momentum (Cohen, 1967b, 1968b; Komar, 1962) and to the rotational kinetic energy. In this paper, these quantities are computed and expressed in a form which is useful for astrophysical calculations.

2. Field Equations

In this section we treat slowly rotating stars. These are stars for which the 'centrifugal'

* Cohen (1967a, 1967b, 1968a, 1968b), Brill and Cohen (1966), Cohen and Brill (1968).

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force acting on any element of the star is small compared to the gravitational force and for which the velocity of any element of the star is small compared to that of light. Such conditions are fulfilled for a typical neutron star of radius 13 km, mass 1.3×10^{33} gm, and rotational period 33 msec or more.

To describe such a star via general relativity, one must consider a line element more general than the familiar Schwarzschild line element

$$ds^2 = -A^2\,dt^2 + B^2\,dr^2 + C^2\,d\theta^2 + E^2(d\phi - \Omega\,dt)^2, \tag{1}$$

where A, B, C, E and Ω are functions of r and θ only (Cohen and Brill, 1968). For slow rotation, the metric (1) can be put in the form

$$ds^2 = -e^{\nu}\,dt^2 + e^{\lambda}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta(d\phi - \Omega\,dt)^2, \tag{2}$$

with ν, λ and Ω functions of r only (to first order in Ω). Three of Einstein's field equations are identical with those for a non-rotating star

$$m_r = 4\pi r^2 \varrho, \tag{3}$$

$$c^2\nu_r/2 = \frac{G(m + 4r^3 p/c^2)}{r(r - 2Gm/c^2)}, \tag{4}$$

$$p_r = -(\varrho + p/c^2)\,c^2\nu_r/2, \tag{5}$$

(Cohen *et al.*, 1969), where the subscript r denotes partial differentiation with respect to r. Here m is the mass, ϱ the density, p the pressure, c the speed of light, and r a radial parameter. In addition to these three equations, there is one additional equation describing a slowly rotating star

$$[e^{-(\lambda+\nu)/2}r^4\Omega_r]_r = -16\pi r^4\,e^{(\lambda-\nu)/2}(\varrho + p/c^2)(\omega - \Omega)\,G/c^2, \tag{6}$$

where ω is the angular velocity of the star and Ω is the angular velocity of the inertial frames along the rotation axis. (The full set of equations describing rotating stars, valid for strong gravitational fields and fast rotation is given by Cohen and Brill (1968).) The angular velocity of inertial frames can be determined, e.g., by measuring the angular velocity of the axis of a gyroscope. Off the rotation axis of the star, the expression for the angular velocity of inertial frames is more complicated than it is on the rotation axis (Cohen, 1968a). Unlike in Newtonian mechanics, the angular velocity of inertial frames in the vicinity of a rotating body does not vanish (relative to the inertial frames far from the body) since according to general relativity rotating bodies drag along inertial frames (Cohen, 1965, 1967a; Thirring, 1918; Brill and Cohen, 1966; Cohen and Brill, 1968). Such effects are in accordance with Mach's idea that inertial properties of space are influenced by the distribution of matter (stress-energy) in the universe. For a discussion of such effect and the effect of other matter in the universe see, e.g., Cohen (1968a) and the references cited there.

At first sight it may seem that the dragging along of inertial frames by rotating bodies is of interest only in philosophical discussions of Mach's principle (Mach, 1883). However, in the following sections, it will be shown that the rotation of the inertial frames contributes to quantities of physical interest such as: angular momentum and rotational energy of stars.

3. Conservation Laws

It is well known that, in Newtonian mechanics and special relativity, symmetries give rise to conserved quantities. For example, momentum and energy conservation are consequences of space and time translation invariance respectively; angular momentum conservation is a consequence of rotational invariance.

Such symmetries can be exhibited mathematically by showing that quantities remain unchanged when one transforms from one point to another. If a quantity such as a scalar remains unchanged when it is transported from one point to another, the set of infinitesimal transformations which generate the transformation can be shown to form a group (Spain, 1960). Because of this, once a symmetry group is found, the machinery of group theory can be used to find conserved quantities.

Similarly, in general relativity, symmetry groups give rise to conserved quantities. The group of transformations in general relativity which leave a given metric unchanged is known as an isometry group. The isometry group associated with this symmetry is generated by a Killing vector ξ_μ which satisfies the equation

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \tag{7}$$

(Eisenhart, 1964; Schild, 1967).

A reader not familiar with Killing vectors may be wondering how a vector can generate a transformation. As an example, consider the Killing vector associated with space translation invariance in flat space. There are three such vectors, one along each axis, but for definiteness consider the Killing vector along the x-axis $\xi_\mu=(0, 1, 0, 0)$. The operator $\xi^\mu \partial_\mu$ generates an infinitesemal transformation along the x-axis, i.e., $f(x+\varepsilon)=f(x)+\varepsilon\partial_x f(x)$. Similarly, the operator $e^{\varepsilon\xi^\mu}\partial_\mu$ generates a finite transformation

$$e^{\varepsilon\partial_x} f(x) = \sum_{n=0}^{\infty} \varepsilon^n \frac{\partial_x^n}{n!} f(x) = f(x+\varepsilon),$$

the second equality is just a Taylor expansion of $f(x+\varepsilon)$. Note that the operator $e^\varepsilon \delta_x$ transforms $f(x)$ into $f(x+\varepsilon)$. Because of the close connection between vectors and generators of transformations, basis vectors are often denoted by partial derivatives ∂_μ and the vector A^μ by $A=A^\mu\partial_\mu$. For a detailed discussion of generators of transformation see, e.g., Chevalley (1950).

When combined with the conservation law of general relativity

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{8}$$

the Killing vector ξ^μ gives rise to a conserved quantity. (Equation (8) is often called a conservation law since in the small velocity and Newtonian limit, it reduces to conservation laws, e.g., for $\mu=0$, it reduces to the familiar equation of continuity

(conservation of mass)

$$\nabla \cdot (\varrho V) = -\, \partial_t \varrho \,,$$

where ϱ is the mass density and V its velocity. Similar results are obtained for $\mu = i$.)

Contraction of the Killing vector ξ_μ with Equation (8) and integration over all space-time yields

$$0 = \int \xi_\mu T^{\mu\nu};_\nu \, \mathrm{d}V_4 \,. \tag{9}$$

Since ξ_μ is a Killing vector satisfying Equation (7), the integrand of Equation (9) becomes a perfect divergence, i.e., $\xi_\mu T^{\mu\nu};_\nu = (\xi_\mu T^{\mu\nu});_\nu - \xi_{\mu;\,\nu} T^{\mu\nu}$. The second term vanishes since the stress energy tensor $T^{\mu\nu}$ is symmetric and $\xi_{\mu;\,\nu}$ is antisymmetric, Equation (7). By using the four-dimensional form of the divergence theorem (Cohen, 1968a), the integral (9) can be transformed to an integral over the three-dimensional surface of the four-dimensional space-time. If the source is bounded in space or falls off sufficiently rapidly at spacial infinity, we find that the integral

$$I = \int_\Sigma \xi_\mu T^{\mu\nu} \, \mathrm{d}\sigma_\nu \tag{10}$$

is independent of the space-like surface and, consequently, is conserved. For the mathematical details see, e.g., Cohen (1968b), (1967a); Komar (1962).

In the following sections, Equation (10) will be used to find an expression for the angular momentum of a rotating star.

4. Angular Momentum

In general relativity, the angular momentum of a body is constant if no external forces act and if the body is sufficiently symmetric that it does not emit gravitational radiation as it rotates. This constant of the motion can be obtained from Equation (10) if the Killing vector associated with the rotation is given. In terms of an orthonormal basis along the $\mathrm{d}t$, $\mathrm{d}r$, $\mathrm{d}\theta$, and $\mathrm{d}\varphi - \Omega\, \mathrm{d}t$ direction, this Killing vector is given by

$$\xi^\mu = [0, 0, 0, E] \,. \tag{11}$$

For a uniformly rotating body, the components of the 4-velocity U^μ are

$$U^\mu = [1, 0, 0, E(\omega - \Omega)/A] \, K \,, \tag{12}$$

where

$$K = (1 - E^2 (\omega - \Omega)^2 A^{-2})^{-1/2} \,. \tag{13}$$

On a space-like surface $t = \mathrm{const}$, the 3-volume element $\mathrm{d}\sigma_\nu$ becomes

$$\mathrm{d}\sigma_0 = BCE \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi ;$$

and, consequently, only the T^{03} component of the stress-energy tensor (here and in the sequel we use geometrized units (Harrison *et al.*, 1965) to simplify the calculations – i.e. mass in centimeter, radius in centimeters, time in centimeters, etc.)

$$T^{\mu\nu} = (\varrho + p)\, U^{\mu} U^{\nu} + p g^{\mu\nu} \tag{14}$$

i.e.,

$$T^{03} = (\varrho + p)\, E(\omega - \Omega)\, A^{-1} K^2 \tag{15}$$

enters Equation (10) which yields the angular momentum of the star

$$J = \int\limits_{t=\mathrm{const}} (\varrho + p)\, A^{-1} B C E^3 K^2 (\omega - \Omega)\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{16}$$

when Equation (16) is integrated across the star. It is interesting to observe that the pressure and the gravitational potential contribute to the angular momentum (16). For a treatment of more general surfaces see, e.g., Cohen (1968b) and the references cited there. The expression (16) reduces to a simple form for a slowly rotating star:

$$J = (8\pi/3) \int (\varrho + p)\, e^{-\nu/2}\, e^{\lambda/2} r^4 (\omega - \Omega)\, \mathrm{d}r\,, \tag{17}$$

after integrating over angles. Here $\nu/2$ reduces to the Newtonian gravitational potential in the Newtonian limit. The general relativistic expressions for the angular momentum (16) and (17) have all of the usual properties of angular momentum in Newtonian mechanics. Unfortunately, the concept of moment of inertia is not as useful in general relativity as it is in Newtonian mechanics since a body rotating rigidly relative to an observer at infinity will not necessarily be rotating rigidly relative to local inertial frames. Stars can rotate rigidly and differentially at the same time depending on what the rotation is measured relative to.

5. Rotational Energy

Besides nuclear energy and pulsational energy, the rotation of a star can store large amounts of energy. It has been suggested (Finzi and Wolf, 1968) that the rotation of a neutron star in the Crab nebulae is the source for the energy emitted by the Crab. A loss of rotational energy would exhibit itself as a decrease in rotation rate of the star. These authors suggest no mechanism for slowing down the star and such questions will not be treated here. We will concern ourselves with obtaining the general relativistic expression for the rotation energy of a star.

In the region at large distances from a star of any shape, the metric reduces to the Schwarzschild metric if there is no gravitational radiation being emitted. Consequently, the gravitational mass of such a star equals the Schwarzschild mass. The rotational kinetic energy of a star is equal to the total mass of the rotating star minus the Schwarzschild mass of the same star when it is not rotating. This rotational energy can be obtained by considering the line element for a rotating axially symmetric star (1), and the field equation $G^{00}=8\pi T^{00}$ which takes the form (Cohen and Brill, 1968)

$$-8\pi T^{00} = B^{-1}C^{-1}\left[(C_r/B)_r + (B_\theta/C)_\theta + E^{-1}(CE_r/B)_r + E^{-1}(BE_\theta/C)_\theta\right] + (E\Omega_r/2AB)^2 + (E\Omega_\theta/2AC)^2 . \quad (18)$$

For an axially symmetric stellar model, the coordinates can be chosen in such a way that the metric coefficients take the form (to second order in Ω) (Hartle and Thorne, 1968)

$$\begin{aligned} A &= \bar{A}(r) + f_1(r) + f_2(r)\, y_2^0(\theta), \\ B &= (1 - 2m(r)/r)^{-1/2} + g(r)\, y_2^0(\theta), \\ C &= r + h(r)\, y_2^0(\theta), \\ E &= C\sin\theta , \\ \Omega &= \Omega(r). \end{aligned} \quad (19)$$

When these relations (19) are substituted into Equation (18) and the result integrated with respect to θ, there results to second order in ω:

$$m_r = \int T^{00} r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + r^4\Omega_r^2/12A^2B^2 . \quad (20)$$

The perturbations f_2, g, and h drop, to second order in Ω because of orthogonality of spherical harmonics. Note that the gravitational mass m depends on the motion of inertial frames Ω as well as on the energy density T^{00}.

The rotational kinetic energy can be obtained from this expression (20) when the expression for the four-velocity (12) is substituted into the expression for the energy density (14)

$$T^{00} = \varrho + (\varrho + p)\, E^2A^{-2}(\omega - \Omega)^2 , \quad (21)$$

and this Equation (21) is substituted into Equation (20) yielding

$$m_r = \int \left[\varrho + (\varrho + p)\, E^2A^{-2}(\omega - \Omega)^2\right] r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi + r^4\Omega_r^2/12A^2B^2 . \quad (22)$$

An integral equation for the rotational kinetic energy can be obtained if we substitute into Equation (21)

$$m = \bar{m} + m_2 , \quad (23a)$$

and

$$\varrho = \bar{\varrho} + \varrho_2, \quad \text{etc.}, \quad (23b)$$

where barred quantities represent unperturbed quantities and m_2, etc. are perturbations yielding from Equation (22)

$$\bar{m}_r = \int \bar{\varrho} r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \quad (24)$$

and

$$(m_2)_r = \int [\varrho_2 + (\bar{\varrho} + \bar{p}) E^2 A^{-2} (\omega - \Omega)^2] \, r^2 \sin\theta \, d\theta \, d\phi + r^4 \Omega_r^2 / 12 A^2 B^2 , \tag{25}$$

where Equation (24) is the familiar equation for the gravitational mass of the unperturbed (non-rotating) stellar model, while Equation (25) is an integral equation for the rotational kinetic energy m_2. Equation (25) can be solved if the density perturbation ϱ_2 is expressed in terms of unperturbed quantities and the kinetic energy m_2. This will be done below by using entropy and baryon conservation.

The entropy change dS is related to the change of internal energy dU and the work done by the system p dV via the well-known relation (Zemansky, 1957)

$$T \, dS = dU + p \, dV , \tag{26}$$

where T is the temperature, p is the pressure, and V is the volume of the system. For convenience, consider N particles occupying a small volume V at an arbitrary point in the system and assume that the various quantities are constant throughout the small volume V (the volume V can be made small enough so that nothing changes much within it). When the expression for the energy density

$$\varrho = U/V \tag{27}$$

is substituted into Equation (26), we obtain

$$T \, dS/V = (\varrho + p)(dV/V) + d\varrho . \tag{28}$$

If the number of baryons N in the proper volume V is not influenced by stellar rotation and since the baryon number density n is related to the volume V by

$$n = N/V , \tag{29}$$

the change in baryon number density depends only on the change in proper volume

$$dn/n = - \, dV/V . \tag{30}$$

Substituting this into Equation (28) and letting dS vanish (for adiabatic process) we find that

$$d\varrho/(\varrho + p) = dn/n . \tag{31}$$

From this (31), we obtain the relation between the energy density perturbation ϱ_2 and the baryon number density perturbation n_2 in the form

$$\varrho_2/(\varrho + p) = n_2/n . \tag{32}$$

Equation (32) reduces the complicated problem of finding the difference in energy density between the rotating and non-rotating configurations ϱ_2, to the simpler problem of finding the difference in baryon number densities n_2. The latter problem is simpler since the number of baryons is the same for the rotating and non-rotating configurations and, consequently, only the volume element must be transformed.

There are two contributions to n_2: (1) Lorentz contraction and (2) the change in geometry because of the change in gravitational mass (rotational kinetic energy contributes to the gravitational mass). In finding baryon number density n_l in the frame of the metric (1), use can be made of the relation

$$n\,\mathrm{d}V = n_l\,\mathrm{d}V_l \tag{33}$$

between the number density n in the rest frame of the matter and that in the frame of the metric (1).

Since the 4-volume element is invariant with respect to coordinate transformations, the 3-volume elements are related

$$\sqrt{|g|}\,\mathrm{d}^4x = A\,\mathrm{d}V_l\,\mathrm{d}t = \mathrm{d}V\,\mathrm{d}\tau\,, \tag{34}$$

which implies that the baryon number densities are related via

$$n = n_l\,\mathrm{d}\tau/A\,\mathrm{d}t\,. \tag{35}$$

The relation between the proper time τ in the rest frame of the matter and the coordinate time t of the metric (1) can be found in standard texts (see, e.g., Landau and Lifshitz, 1962; Møller, 1960).

$$\mathrm{d}\tau/A\mathrm{d}t = |(1-\gamma_k v^k(-g_{00})^{-1/2})^2 - \gamma_{ij}v^i v^j(-g_{00})^{-1}|^{1/2}\,, \tag{36}$$

where the 3-velocity vector $v^i = \mathrm{d}x^i/\mathrm{d}t$ is given by

$$v = |0, 0, \omega|\,, \tag{37}$$

while γ_k and γ_{ij} are defined by

$$\gamma_k = g_{0k}/(-g_{00})^{1/2} \tag{38}$$

and

$$\gamma_{ij} = g_{ij} + \gamma_i\gamma_j\,, \tag{39}$$

where Latin letters have the range 1, 2, 3 when summed. Substitution of the metric (1) into Equations (38) and (39) and these into Equation (36) yields

$$\mathrm{d}\tau/A\,\mathrm{d}t = 1 - E^2A^{-2}(\omega-\Omega)^2/2 \tag{40}$$

to second order in ω. When Equation (40) is substituted into Equation (35), we obtain the relation between the baryon number density in the rest frame of the matter n and that in the frame of the metric (1) with respect to which the star is rotating n_l.

The baryon number density n_l defined in Equation (33a), takes the form

$$n_l = \mathrm{d}N/\mathrm{d}V_l = \mathrm{d}N/(BCE\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi)\,. \tag{41}$$

Because the rotational energy contributes to the gravitational mass, the volume element $\mathrm{d}V_l$ is not equal to the volume element $\mathrm{d}\bar{V}$ of the same star when not rotating. The relation between the baryon number density n_l and the baryon number density of the non-rotating star $\bar{n}$ can be obtained by expanding B, C, and E in Equation (41)

giving
$$n = \bar{n}[1 - B_2\bar{B}^{-1} - C_2\bar{C}^{-1} - E_2\bar{E}^{-1}]. \tag{42}$$

A substitution of Equations (42) and (40) in Equation (33) yields the baryon number density in the frame of the matter (expressed in terms of unperturbed (barred) quantities and perturbations):

$$n_t = \bar{n}[1 - B_2\bar{B}^{-1} - C_2\bar{C}^{-1} - E_2\bar{E}^{-1} - E^2A^{-2}(\omega - \Omega)^2/2]. \tag{43}$$

The baryon number density perturbation is obtained by comparing Equation (43) and

$$n = \bar{n} + n_2. \tag{44}$$

Substitution of the resultant value of m_2 into Equation (32) yields

$$\varrho_2 = (\varrho + p)[-E^2A^{-2}(\omega - \Omega)^2/2 - B_2\bar{B}^{-1} - C_2\bar{C}^{-1} - E_2\bar{E}^{-1}]. \tag{45}$$

This equation can, in turn, be substituted into Equation (25) yielding an integral-differential equation for the rotational energy m_2 as

$$\begin{aligned} m_2)_r = \int r^2 \sin\theta \, d\theta \, d\phi \, [\bar{\varrho} + \bar{p}] \\ \times [E^2A^{-2}(\omega - \Omega)^2/2 - B_2\bar{B}^{-1} - C_2\bar{C}^{-1} - E_2\bar{E}^{-1}] \\ + r^4\Omega_r^2/12A^2B^2. \end{aligned} \tag{46}$$

Integration with respect to θ, and ϕ reduces Equation (46) to the differential equation

$$\begin{aligned} (m_2)_r = \frac{8\pi}{3}(\bar{\varrho} + \bar{p})\, r^4A^{-2}(\omega - \Omega)^2/2 \\ - 4\pi(\bar{\varrho} + \bar{p})\, rm_2B^2 + r^4\Omega_r^2/12A^2B^2. \end{aligned} \tag{47}$$

This Equation (47) can be solved for m_2. To do this we can use the relation

$$8\pi r(\bar{\varrho} + \bar{p})\, e^{\bar{\lambda}} = (\bar{\lambda} + \bar{\nu})_r \tag{48}$$

which can be obtained by substituting Equation (19) into Equations (2) and (3) and combining the latter two equations.

Substitution of Equation (48) into Equation (47) yields

$$[e^{(\bar{\lambda}+\nu)/2}m_2]_r = \frac{4\pi}{3}(\bar{\varrho} + \bar{p})\, r^4 e^{(\bar{\lambda}-\nu)/2}(\omega^2 - 2\Omega\omega + \Omega^2) + r^4\Omega_r^2/12AB. \tag{49}$$

Integration of Equation (49) yields an expression for the rotational kinetic energy

$$m_2 = (4\pi\omega^2/3)\int_0^R (\bar{\varrho} + p)\, r^4 e^{(\bar{\lambda}-\nu)/2}[(1 - \Omega/\omega)^2]\, dr + \int_0^\infty r^4\Omega_r^2(12AB)^{-1}\, dr, \tag{50}$$

where R is the radius of the star. In the Newtonian limit, this reduces to

$$m_2 = (4\pi\omega^2/3) \int_0^R \varrho r^4 \, \mathrm{d}r \tag{51}$$

the familiar expression for the rotational kinetic energy of a slowly rotating spherical body in Newtonian mechanics. The second integral in Equation (50) is difficult to evaluate with a computer because of its infinite range of integration. This difficulty can be circumvented, however, since an analytic solution is known in the region exterior to the star (Cohen, 1965, 1967a; Brill and Cohen, 1966; Cohen and Brill, 1968). To evaluate the integral we only need the solution to first order in ω:

$$\begin{aligned} A^2 = B^{-2} &= 1 - 2m/r, \\ \Omega &= 2J/r^3, \end{aligned} \tag{52}$$

giving

$$\int_R^\infty r^4 \Omega_r^2 (12AB)^{-1} \, \mathrm{d}r = J^2/R_3^3. \tag{53}$$

An expression for the rotational kinetic energy which is convenient for numerical calculations (e.g., for rotating neutron star models) is obtained by substituting Equation (53) into Equation (50) in the form

$$\begin{aligned} m_2 = (4\pi\omega^2/3) \int_0^R (\varrho + p) \, r^4 B A^{-1} [(1 - \Omega/\omega)^2] \, \mathrm{d}r \\ + \int_0^R r^4 \Omega_r^2 (12AB)^{-1} \, \mathrm{d}r + J^2/R^3. \end{aligned} \tag{54}$$

Note that the pressure, red shift, the change in volume element due to the curvature of space, and the motion of inertial frames contribute to this expression for the rotational kinetic energy of a slowly rotating stellar model. Numerical calculations for neutron star models are in progress and will be presented elsewhere (Cohen and Cameron, 1969).

6. Discussion

To facilitate the calculations performed in the preceding sections, geometrized units (distances in cm, time in cm, etc.) were used. However, for astrophysical applications, it is usually more convenient to use results which are expressed in familiar units. In standard cgs units the angular momentum is given by

$$J = (8\pi/3) \int_0^R (\varrho + p/c^2) \, B A^{-1} (\omega - \Omega) \, r^4 \, \mathrm{d}r, \tag{55}$$

and the rotational kinetic energy is given by

$$E_{\text{rot}} = (4\pi/3) \int_0^R (\varrho + p/C^2)\, r^4 B A^{-1} [(\omega - \Omega)^2]\, \mathrm{d}r + \int_0^R \mathrm{d}r\, r^4 \Omega_r^2 C^2 (12ABG)^{-1} + GJ^2/R^3C^2, \tag{56}$$

where ϱ is the density, p the pressure, c the speed of light, G the gravitational constant, R the radius of the star, J its angular momentum. The quantities A^2 and B^2 are components of the metric (1) which can be found by solving Equations (3)–(5). The angular velocity ω of the star is given relative to an observer at infinity while the angular velocity of inertial frames along the rotation axis Ω is given relative to the same observer and can be found by integrating Equation (6). It is of interest to note that both the angular momentum J and the kinetic energy E_{rot} contain contributions from the pressure and from the rotation of inertial frames.

According to general relativity (and the scalar tensor theory also), rotating bodies drag along the gravitational field and the inertial frames. For rotating dense stellar models, the rotating gravitational field can make a non-negligible contribution to the rotational energy. Such results have been found for neutron star models (Cohen and Cameron, 1969). The last term in Equation (56) is the contribution to the rotation energy from the gravitational field *outside the star*.

When Equations (55) and (56) are applied to neutron stars (Cohen and Cameron, 1969), they give the angular momentum and the rotational kinetic energy stored in such stars.

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